NORTH SYDNEY BOYS HIGH SCHOOL

Mathematics Extension 1

2024 Assessment Task 3

General Instructions:

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- For questions in Section B, show all relevant mathematical reasoning and/or calculations
- Use a new booklet for each question
- Write your student number and tick teacher name on **everything**

Teacher

(Please tick)

○ Mr Berry
○ Ms Cai
○ Mr Ireland
○ Ms Lee
○ Ms Moss
○ Mr Umakanthan
○ Dr Vranešević

Student Number:

4								

Marker Use Only:

Question	MC	11	12	13	14	Total	%
Mark	/10	/15	/15	/15	/15	/70	

Section A – Multiple Choice (10 marks)

1. Given $\underset{\sim}{u} = 2\underset{\sim}{i} - 6\underset{\sim}{j}$ and $\underset{\sim}{v} = -\underset{\sim}{i} + 5\underset{\sim}{j}$, $\underset{\sim}{u} - \underset{\sim}{v}$ is equivalent to:

   A. $\underset{\sim}{i} - \underset{\sim}{j}$

   B. $3\underset{\sim}{i} - \underset{\sim}{j}$

   C. $\underset{\sim}{i} - 11\underset{\sim}{j}$

   D. $3\underset{\sim}{i} - 11\underset{\sim}{j}$

2. Let α, β and γ be the roots of $x^3 + px^2 + q = 0$.
   Express $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma}$ in terms of p and q

   A. pq

   B. $-pq$

   C. $-\frac{p}{q}$

   D. $\frac{p}{q}$

3. Harry projects an arrow at an angle of 60° to the horizontal with an initial velocity of 30 ms^{-1}. What is the horizontal speed of the arrow?

   A. $60\ \text{ms}^{-1}$

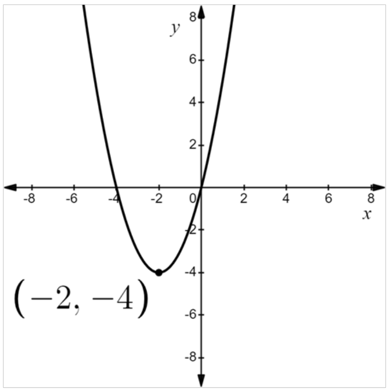
   B. $\frac{30}{\sqrt{3}}\ \text{ms}^{-1}$

   C. $30\sqrt{3}\ \text{ms}^{-1}$

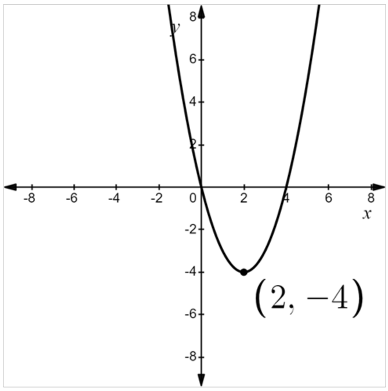
   D. $15\ \text{ms}^{-1}$

4. Given $f(x) = x^2 - 4$ and $g(x) = |-x - 2|$, which graph represents $f\big(g(x)\big)$?

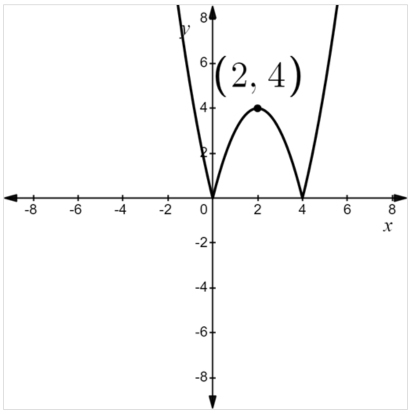
A.



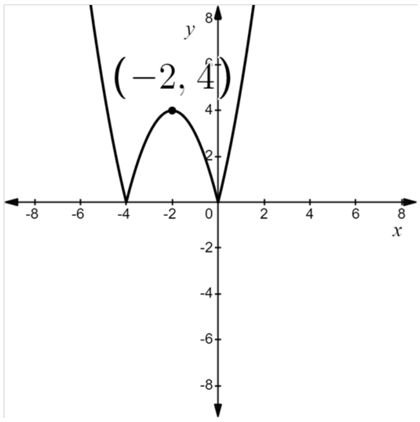
B.



C.



D.



5. If $\cos x \ = -\frac{4}{5}$ and $\frac{\pi}{2} \leq x \leq \pi$ then $\tan 2x$ is equal to:

A. $-\frac{24}{7}$

B. $\frac{24}{7}$

C. $\frac{12}{7}$

D. $-\frac{12}{7}$

6. Given $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{v} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$, $proj_{\underset{\sim}{u}}\underset{\sim}{v}$ is:

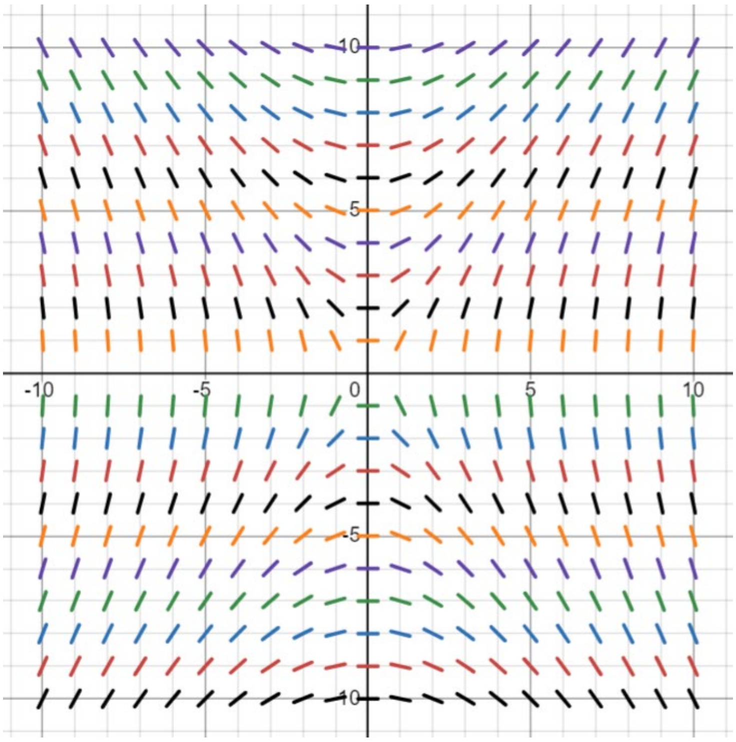
A. $-\frac{16}{13}\underset{\sim}{i} + \frac{32}{13}\underset{\sim}{j}$

B. 8

C. $\frac{8}{13}$

D. $\frac{16}{13}\underset{\sim}{i} + \frac{24}{13}\underset{\sim}{j}$

7. Which of the following differential equations could be represented by the slope field drawn below?



A. $\frac{dy}{dx} = \frac{2x}{y}$

B. $\frac{dy}{dx} = \frac{2y}{x}$

C. $\frac{dy}{dx} = \frac{x^2}{y}$

D. $\frac{dy}{dx} = \frac{x}{y^2}$

8. A curve C has parametric equations $x = \cos^2 t$ and $y = 4\sin^2 t$ for $t \in R$.

What is the Cartesian equation of C?

A. $y = 1 - x$ for $0 \leq x \leq 1$

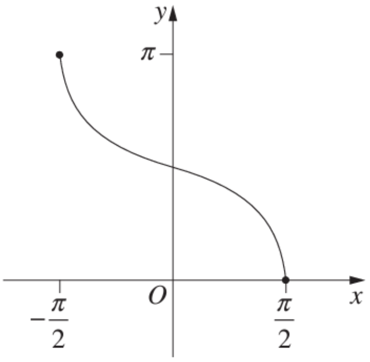
B. $y = 4 - 4x$ for $x \in R$

C. $y = 1 - x$ for $x \in R$

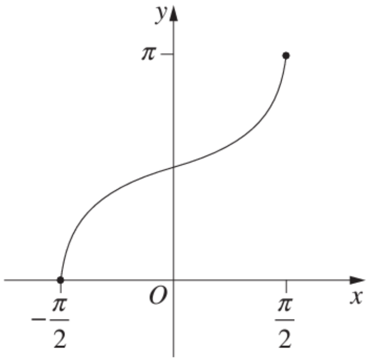
D. $y = 4 - 4x$ for $0 \leq x \leq 1$

9. Which graph best represents $y = \cos^{-1}(-\sin x)$, for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

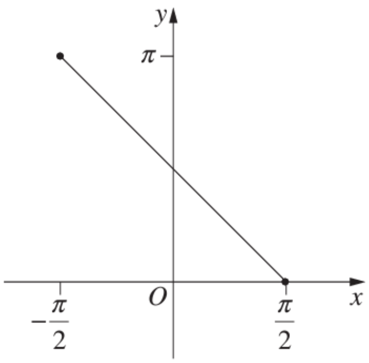
A.



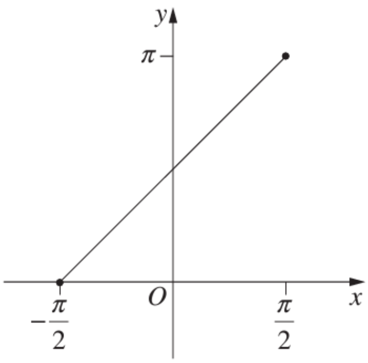
B.



C.

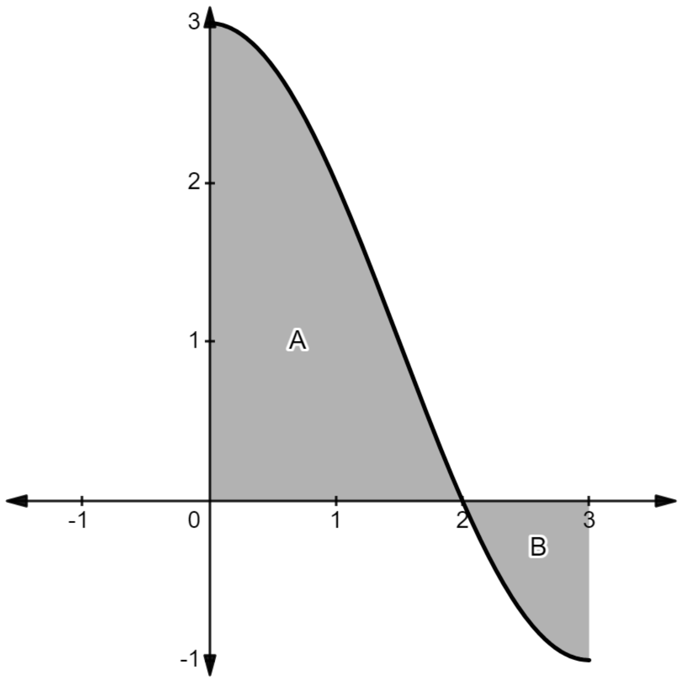


D.



10.

The graph of $y = 2\cos\frac{\pi x}{3} + 1$ is shown



Region A has an Area of $\frac{3\sqrt{3}}{\pi} + 2$ units

Region B has an Area of $\frac{3\sqrt{3}}{\pi} - 1$ units

Using this information, evaluate $\int_{-1}^{3} \frac{3}{\pi} \cos^{-1}\frac{x-1}{2}\, dx$

A. 3

B. 6

C. $\frac{6\sqrt{3}}{\pi} + 1$

D. $11 - \frac{6\sqrt{3}}{\pi}$

Section B – Questions 11-15 (60 marks total)

Question 11 (15 marks)

(START A NEW BOOKLET)

a) Solve for x: 2

$$\frac{6}{x-2} \geq 3$$

b) Find the exact value of: 2

$$\sin\left(2\cos^{-1}\left(\frac{2}{3}\right)\right)$$

c) Find, in simplest terms, the coefficient of the x^6 term in the expansion of: 2

$$\left(2x^2 - \frac{1}{3x}\right)^9$$

d) How many numbers greater than 8000 can be formed from the digits 1, 2, 4, 6, 9 if no digit is repeated? 2

e) Use $t = \tan\frac{\theta}{2}$ to solve 2

$$\sin\theta + \cos\theta = -\frac{1}{4} \text{ for } -\pi \leq \theta < \pi$$

f) Given $y = f(x)$ where $f(x) = (x-1)^2 - 4$. Sketch the following curves on separate graphs, each at least one-third of a page in size:

(i) $y = -f(2-3x)$ 3

(ii) $y = |f(|x|)|$ 2

Question 12 (15 marks)

(START A NEW BOOKLET)

a) Using the substitution $u^2 = x + 1$ where $u > 0$ to find: 3

$$\int_0^3 \frac{x+2}{\sqrt{x+1}}\,dx$$

b) Given $f(x) = \sin^{-1} x$ and $g(x) = \cos^{-1} x$

(i) Sketch $f(x)$ and $g(x)$ on the same set of axes 2

(ii) Hence, state the value of $f(x) + g(x)$ for $-1 \leq x \leq 1$ and explain the significance of this finding. 1

c) By methods of induction, prove that $3^{2n+2} - 8n - 9$ is divisible by 64 for all integers $n \geq 1$ 3

d) Show that if 19 distinct numbers are chosen from the sequence 1, 4, 7, 10, …, 100 there must be two of them whose sum is 104. 2

e) Given $4x^4 + 8x^3 + 3x^2 - 2x - 1 = 0$

(i) Express in the form $(ax^2 + bx)^2 - (cx + d)^2 = 0$, where a, b, c and d are positive integers, and state the values of a, b, c and d. 2

(ii) Hence solve the equation for x 2

Question 13 (15 marks)

(START A NEW BOOKLET)

a) Given $f(x) = x\cos^{-1}x - \sqrt{1-x^2}$

(i) Find $f'(x)$ 2

(ii) Hence, evaluate: 2

$$\int_0^1 \cos^{-1}x \; dx$$

b) Let T be the temperature inside B15 at time t and let A be the constant outside air temperature. Newton's law of cooling states that the rate of change of the temperature T is proportional to $(T - A)$. It can be shown that $T = A + Ce^{kt}$ where C and k are constants satisfies Newton's law of cooling.

The outside air temperature is $15°\ C$ and Year 12 come in from lunch and open all the windows. The temperature inside drops from $25°C$ to $21°C$ over a period of half an hour.

(i) Find the values of C and k 2

(ii) How much longer would it take the temperature to drop to 16°C? Give your answer to the nearest minute. 1

c) A coach is watching a gridiron player from a point O on the sideline. He looks directly at the player without taking his eyes off him. The player starts at position A with position vector $\overrightarrow{OA} = 25\underset{\sim}{i} + 30\underset{\sim}{j}$. The player runs in a straight line with a constant speed. After 2 seconds, the player is at position B with position vector $\overrightarrow{OB} = 19\underset{\sim}{i} + 33\underset{\sim}{j}$. The coach watches the player run for 10 seconds in total starting from position A and finishing at position C.

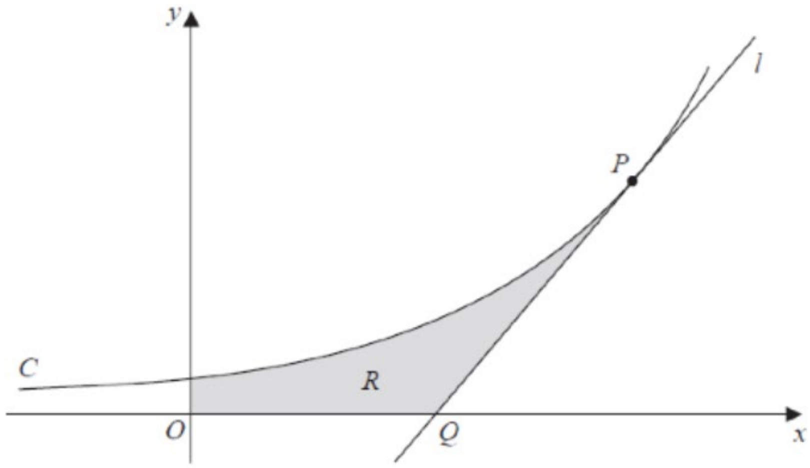
(i) After 10 seconds, what is the position vector $\overrightarrow{OC}$ of the player relative to the coach? 2

(ii) Through what angle does the coach turn his head in order to watch them for the full 10 seconds? Answer to the nearest minute. 2

Question 13 (cont.)

d) The diagram shows a region bound by the curve C with the equation $y = 3^x$ the line l and the x-axis.

The x-co-ordinates of points P and Q are 2 and $(2 - \frac{1}{\ln 3})$ respectively.



A solid is created when the region R is rotated around the x-axis. Find the volume of the solid formed. 4

Question 14 (15 marks)

(START A NEW BOOKLET)

a) Solve:

$$6\sin^2 x + 4\sin x\cos x - 3\sin x = 0 \quad \text{for} \quad 0 \le x \le \pi$$ 3

b) Given

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

(i) Show that $y = f(x)$ has no stationary points. 2

(ii) Given that $y = \pm 1$ are horizontal asymptotes, sketch the curve. 2

(iii) For $k > 0$, consider the area enclosed by the curve, the lines $y = 1, x = 0$ and $x = k$. Show that this area can be expressed in the form: 2

$$\ln\left(\frac{2e^k}{e^k + e^{-k}}\right)$$

(iv) Hence, justify why for all values of k, the area found in part (iii) is always less than $\ln 2$. 1

c) A particle is projected from a point O with speed $80\ \text{ms}^{-1}$ at an angle of elevation α, where $\tan\alpha = \frac{5}{12}$. Two seconds later, a second particle is projected from O and it collides with the first particle one second after leaving O. Let β be the initial angle of projection for the second projectile.
Let $g = 10\ \text{ms}^{-2}$.

(i) Show that the displacement of the first particle after t seconds is given by the equation:

$$s(t) = 80t\cos\alpha\ \underset{\sim}{i} + (80t\sin\alpha - 5t^2)\underset{\sim}{j}$$ 1

(ii) Find $\tan\beta$ 3

(iii) Find the initial velocity of the second particle 1

END OF EXAMINATION

Multiple Choice Answers

1. D :

   $\mathbf{u}-\mathbf{v}=(2\mathbf{i}-6\mathbf{j})-(-\mathbf{i}+5\mathbf{j})=2\mathbf{i}+\mathbf{i}-6\mathbf{j}-5\mathbf{j}=3\mathbf{i}-11\mathbf{j}$

2. D :

   $\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}=\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}=\frac{-p}{-q}=\frac{p}{q}$

3. D :

   $v\cos 60^\circ=30\times\frac{1}{2}=15\,\text{ms}^{-1}$

4. A :

   Note when $x=-2, g(x)=0, f(x)=-4$.

5. A :

   Either by $\sin^2 x+\cos^2 x=1$ or Pythagoras theorem (adjust for 2nd quadrant, only cos is +ve),
   $\sin x=\frac{3}{5}$ and $\tan x=\frac{-3}{4}\ \Rightarrow \tan 2x=\frac{2\tan x}{1-\tan^2 x}=\frac{2\times\frac{-3}{4}}{1-(\frac{-3}{4})^2}=\frac{-24}{7}$

6. D :

   $\text{proj}_{\mathbf{u}}\mathbf{v}=\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$

   $\mathbf{u}\cdot\mathbf{v}=(2)(-2)+(3)(4)=-4+12=8$

   $\text{proj}_{\mathbf{u}}\mathbf{v}=\frac{8}{13}\mathbf{u}=\frac{8}{13}(2\mathbf{i}+3\mathbf{j})=\frac{16}{13}\mathbf{i}+\frac{24}{13}\mathbf{j}$

7. A:

   Cannot be B as slope is 0 when $x=0$ not undefined.

   Cannot be C or D because slope is +ve when $x<0$ and $y<0$

8. D:

   Since $\cos^2 t+\sin^2 t=1\ \Rightarrow x+\frac{y}{4}=1\ \Rightarrow y=4-4x$

9. D:

   $y=\cos^{-1}(-\sin x)=\pi-\cos^{-1}(\sin x)=\pi-(\frac{\pi}{2}-x)=x+\frac{\pi}{2}$

10. B:

    The integral is equal to the area between the curve and the y axis, which is $A-B+3$

Q11
Tuesday, 6 August 2024 9:37 AM

Section B

Question 11

a) $\frac{6}{x-2} \geqslant 3$

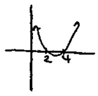
$\frac{2}{x-2} \geqslant 1$

$2x-4 \geqslant (x-2)^2 \quad , x \neq 2$

$2x-4 \geqslant x^2-4x+4$

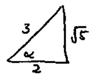
$0 \geqslant x^2-6x+8$

$0 \geqslant (x-4)(x-2)$



$2 < x \leqslant 4$

b) $\sin\left(2\cos^{-1}\left(\tfrac{2}{3}\right)\right)$ Let $\alpha = \cos^{-1}\left(\tfrac{2}{3}\right)$, $\cos\alpha = \tfrac{2}{3}$



$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2\left(\frac{\sqrt{5}}{3}\right)\left(\frac{2}{3}\right)$

$\sin\left(2\cos^{-1}\left(\tfrac{2}{3}\right)\right) = \frac{4\sqrt{5}}{9}$

c) $\left(2x^2 - \frac{1}{3x}\right)^9$ General term

${}^9C_r (2x^2)^r \left(-\tfrac{1}{3}x^{-1}\right)^{9-r}$

$= {}^9C_r 2^r x^{2r} (-1)^{9-r} \left(\frac{1}{3}\right)^{9-r} x^{r-9}$

$= {}^9C_r 2^r (-1)^{9-r} \left(\tfrac{1}{3}\right)^{9-r} x^{3r-9}$

$6 = 3r-9$

$r = 5$

$\Rightarrow$ Coefficient of $r=5$ term

${}^9C_5\, 2^5 (-1)^4 \left(\tfrac{1}{3}\right)^4 = \underline{448}$

${}^9C_5\, 2^5 (-1)^4 \left(\tfrac{1}{3}\right)^4 = \frac{448}{9}$

d) 1, 2, 4, 6, 9 > 8000

4 digits, starting with 9 $9\_\_\_ = 4\times3\times2$ +

5 digits $5! = 5!$

144 numbers

e) $t = \tan\frac{\theta}{2}$ $-\pi \leqslant \theta \leqslant \pi$

$\sin\theta = \frac{2t}{1+t^2}$ $\cos\theta = \frac{1-t^2}{1+t^2}$

$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = \frac{-1}{4}$

$4(1+2t-t^2) = -1-t^2$

$4+8t-4t^2 = -t^2-1$

$0 = 3t^2-8t-5$

$t = \frac{8 \pm \sqrt{64+60}}{6}$

$= \frac{4 \pm \sqrt{31}}{3}$

$t = \frac{4+\sqrt{31}}{3}$ $\quad t = \frac{4-\sqrt{31}}{3}$

$\tan\frac{\theta}{2} = \frac{4+\sqrt{31}}{3}$ $\quad \tan\frac{\theta}{2} = \frac{4-\sqrt{31}}{3}$

$\theta = 2\tan^{-1}\left(\frac{4+\sqrt{31}}{3}\right)$ $\quad \theta = 2\tan^{-1}\left(\frac{4-\sqrt{31}}{3}\right)$

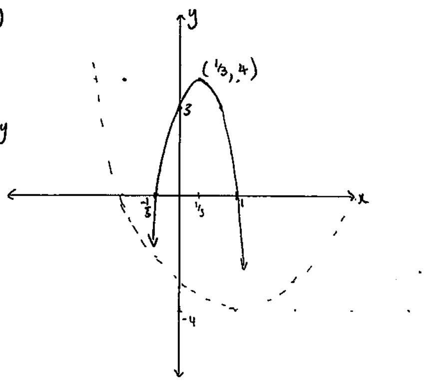
≈ 1.27 $\quad \approx -0.48$

f) $y = (x-1)^2 - 4$

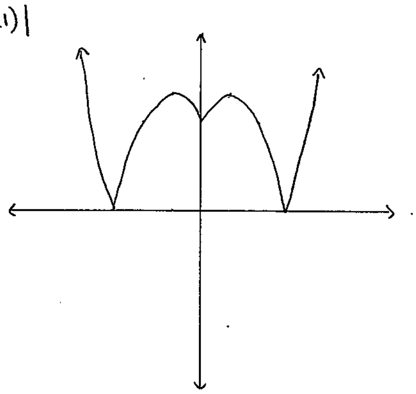
$y = -f(2-3x)$

$y = -f(2-3x)$

- $y = f(x)$ (reflect around x)
- $y = f(x+2)$ shift 2 left
- $y = f(-x+2)$ reflect around y
- $y = f(-3x+2)$ dilate x by 1/3



$y = |f(|x|)|$



Q12

Question 12

a) $u^2 = x+1 \quad u > 0$ $\quad x=0,\ u^2=1,\ u=1 \quad x=3,\ u^2=4,\ u=2$

$\int_0^3 \frac{x+2}{\sqrt{x+1}}\,dx$ $\quad x = u^2-1$, $\frac{dx}{du} = 2u$, $dx = 2u\,du$

$= \int_1^2 \left(\frac{u^2+1}{\sqrt{u^2}}\right) 2u\,du$ | 1 mark for substitution

$= \int_1^2 \frac{2u^3+2u}{u}\,du$

$= \int_1^2 2u^2+2\,du$

$= \left[\frac{2u^3}{3} + 2u\right]_1^2$ | 1 mark for taking the integral.

$= \left[\frac{2(8)}{3} + 2(2)\right] - \left[\frac{2}{3}+2\right]$

$= \frac{16}{3} + 4 - \frac{2}{3} - 2$

$= 2 + \frac{14}{3}$

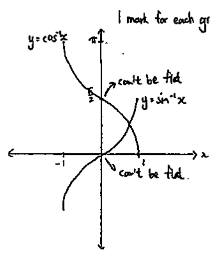
$\frac{20}{3}$ | 1 mark for final answer.

Any mistake for each step is (-1)

C.F.E → carry forward error.

b) $f(x) = \sin^{-1}x \quad g(x) = \cos^{-1}x$

1 mark for each graph.



$f(x) + g(x) = \frac{\pi}{2}$ for $-1 \leqslant x \leqslant 1$

This is because $\sin^{-1}x$ and $\cos^{-1}(x)$ are complementary angles. | 1 mark

For significance, must mention they are complementary angles

c) For $n=1$

$3^{2(1)+2}-8(1)-9 = 3^4-17$
$= 81-17$
$= 64$ which is divisible by 64
$\therefore$ True for $n=1$

Step 1 prove true for $n=1$
1 mark.

Assume true for $n=k$
i.e. $3^{2k+2}-8k-9 = 64m$, $m \in \mathbb{Z}$ (**)

Prove true for $n=k+1$

RTP $3^{2(k+1)+2}-8(k+1)-9 = 64k$, $k \in \mathbb{Z}$

LHS $= 3^{2k+2+2}-8k-8-9$
$= 3^{2k+4}-8k-17$
$= 9\times 3^{2k+2}-8k-17$

From (**) $3^{2k+2} = 64m+8k+9$

$\therefore$ LHS $= 9\times(64m+8k+9)-8k-17$
$= 9\times 64m + 72k + 81 - 8k - 17$
$= 9\times 64m + 64k + 64$
$= 64(9m+k+1)$
which is divisible by 64 since $m, k \in \mathbb{Z}$

Step 2 assume true for $n=k$
&
Step 3 Prove true for $n=k+1$

1 mark

Since true for $n=1$ and true for $n=k+1$ if true for $n=k$ then by PMI true for all $n \geq 1$

Full conclusion
1 mark.

d) 1, 4, 7, 10, ..., 100

Consider pairs which sum to 104
(100, 4) (97, 7) ... (55, 49)

There are 16 such pairs, plus the numbers 1 and 52

1 mark for partically explanation (e.g. didn't mention the leftover 1 and 52)

$\therefore$ If choosing 1, 52 and one number from each pair then by the pigeonhole principle the 19th number must be from one of the pairs already chosen.

2 marks for clear explanation using Pigeonhole Principle.

e) $4x^4+8x^3+3x^2-2x-1=0$

Must state the value of a, b, c and d clearly.
If not, maximum 1 mark.

i) $4x^4+8x^3+4x^2-x^2-2x-1=0$
$(2x^2+2x)^2-(x+1)^2=0$

$a=2,\ b=2,\ c=1,\ d=1$

1 mark for getting at least 2 of them right.
2 marks for all values right
(note a, b, c, d could also all be negative)

$(2x^2+2x-x-1)(2x^2+2x+x+1)=0$
$(2x^2+x-1)(2x^2+3x+1)=0$
$(2x-1)(x+1)(2x+1)(x+1)=0$

$\therefore$ $x=-1$ or $x=\frac{1}{2}$ or $x=-\frac{1}{2}$

1 mark for at least 1 right.
2 marks for all 3 x values right (no extra)

13(a) (i) $f(x) = x\cos^{-1}x - \sqrt{1-x^2}$

$f'(x) = \cos^{-1}x - \frac{x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}$ ✓
$= \cos^{-1}x$

(ii) $\int_0^1 \cos^{-1}x\,dx = \left[x\cos^{-1}x - \sqrt{1-x^2}\right]_0^1$ ✓
$= 0-(0-\sqrt{1})$
$= 1$ ✓

(b) (i) $T = A + Ce^{kt}$
$t=0,\ A=15^\circ C,\ T=25^\circ C$
$\Rightarrow 25 = 15 + C \Rightarrow C=10$ ✓

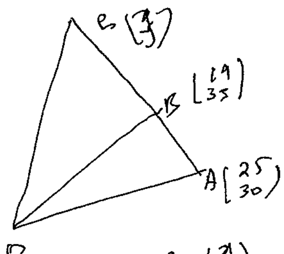
$t = 30$ min
$T = 21^\circ C$
$21 = 15+10e^{30k}$
$k = \frac{\ln 0.6}{30}$ ✓
≈ -0.017

(or)

$t = \frac{1}{2}h$
$k = 2\ln(0.6)$ ✓

(ii) $T = 16^\circ$
$16 = 15 + 10e^{kt}$
$\Rightarrow t = \frac{1}{k}\ln(0.1)$
$k = \frac{\ln(0.6)}{30} \Rightarrow t = 30\frac{\ln(0.1)}{\ln(0.6)} = 135.21\ldots$ ✓
≈ 135 min
or
$k = 2\ln(0.6) \Rightarrow t = \frac{\ln(0.1)}{2\ln(0.6)} = 2.25378\ldots$
$\approx 2h\ 15$ min ✓

13 (c) (i)



Position vector of C $= \vec{OC} = \binom{x}{y}$ ✓

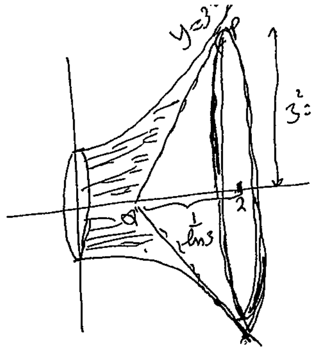
$\binom{x}{y} = 5(\vec{AB}) + \vec{OA}$
$= 5\binom{-6}{5} + \binom{25}{30}$
$= \binom{-5}{45}$ ✓

(ii) $\cos\angle AOC = \frac{\vec{OA}\cdot\vec{OC}}{|\vec{OA}||\vec{OC}|}$

$\vec{OA}\cdot\vec{OC} = \binom{25}{30}\cdot\binom{-5}{45} = 1225$
$|\vec{OA}| = \sqrt{25^2+30^2} = 5\sqrt{61}$
$|\vec{OC}| = \sqrt{(-5)^2+45^2} = 5\sqrt{82}$ ✓

$\therefore \cos\angle AOC = \frac{1225}{25\sqrt{61}\sqrt{82}}$
$\approx 0.6928244\ldots$ ✓
$\Rightarrow \angle AOC = 46^\circ 9'$ (nearest min)

13(c)



Vol. of the required solid (V) = (Vol. of solid generated by rotating $y=3^x$ about the x-axis between $x=0$ and $x=2$)

− (Vol. of solid generated by rotating AP about the x-axis between $x = 2-\frac{1}{\ln 3}$ and $x=2$)

$$\Rightarrow V = \pi\int_0^2 3^{2x}\,dx - (\text{vol. of cone of radius } 9 \text{ and height } \tfrac{1}{\ln 3})$$

$$= \pi\left[\frac{3^{2x}}{2\ln 3}\right]_0^2 - \frac{1}{3}\pi(9)^2\cdot\frac{1}{\ln 3}$$

$$= \pi\left[\frac{81}{2\ln 3} - \frac{1}{2\ln 3}\right] - \frac{27\pi}{\ln 3}$$

$$= \frac{\pi}{\ln 3}\left[40 - 27\right]$$

$$= \frac{13\pi}{\ln 3} \text{ units}^3$$

Q14

Tuesday, 6 August 2024 9:50 AM

Question 14

$\sin x(6\sin x + 4\cos x - 3) = 0$

$\sin x = 0$ or $6\sin x + 4\cos x = 3$

$x = 0, \pi$ $\quad\sqrt{52}\sin(x+\tan^{-1}(\frac{2}{3})) = 3$

$\sin(x+\tan^{-1}(\frac{2}{3})) = \frac{3}{\sqrt{52}}$

$x + \tan^{-1}(\frac{2}{3}) = \sin^{-1}(\frac{3}{\sqrt{52}}), \pi - \sin^{-1}(\frac{3}{\sqrt{52}})$

$x + 0.5880... = 0.42906..., \pi - 0.42906$

$x = -0.158..., 2.124...$

$x = 2.12 \quad 0 \le x \le \pi$

$\therefore x = 0, 2.12, \pi$

- Could also use t-method or similar. Answers in radians

b) i) $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$f'(x) = \frac{(e^x+e^{-x})^2 - (e^x-e^{-x})^2}{(e^x+e^{-x})^2}$ $\quad u = e^x - e^{-x} \quad v = e^x + e^{-x}$ $\quad u' = e^x + e^{-x} \quad v' = e^x - e^{-x}$

$0 = 1 - \frac{(e^x-e^{-x})^2}{(e^x+e^{-x})^2}$ $\quad$ → Also simplifies to $\frac{4}{(e^x+e^{-x})^2}$

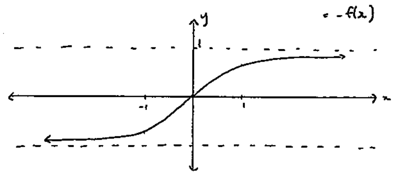
$1 = \frac{e^{2x} - 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}}$

$e^{2x} + 2 + e^{-2x} = e^{2x} - 2 + e^{-2x}$ $\quad$ Since $2 \ne -2$, no stationary points

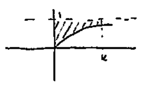
ii) $y = \pm 1$ $\quad f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ $\quad f(-x) = \frac{e^{-x} - e^x}{e^{-x} + e^x}$

$= -f(x)$ $\quad \therefore f(x)$ is odd

$f(1) = 0.76...$



iii)



$$\int_0^k 1 - \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$$

$= [x - \ln(e^x + e^{-x})]_0^k$

$= k - \ln(e^k + e^{-k}) - 0 + \ln 2$

$= \ln e^k + \ln 2 - \ln(e^k + e^{-k})$

$= \ln\left(\frac{2e^k}{e^k + e^{-k}}\right)$

iv) from iii) $A = \ln\left(\frac{2e^k}{e^k + e^{-k}}\right)$

$\lim_{k\to\infty} \ln\left(\frac{2e^k}{e^k + e^{-k}}\right)$ $\quad$ As $k \to \infty$ $e^{-k} \to 0$

$\therefore \ln\left(\frac{2e^k}{e^k + e^{-k}}\right)$ approaches

$\ln\left(\frac{2e^k}{e^k}\right)$

$= \ln 2$

Alternately, consider $\ln\left(\frac{2e^k}{e^k + e^{-k}}\right) = \ln 2 + \ln\left(\frac{e^k}{e^k + e^{-k}}\right)$

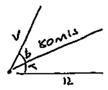
since e^k, e^{-k} are both >0

$e^k < e^k + e^{-k}$

$\therefore \frac{e^k}{e^k + e^{-k}} < 1$

$\therefore \ln\left(\frac{e^k}{e^k + e^{-k}}\right) < 0.$

i)



$\ddot{y} = -10$ $\quad \ddot{x} = 0$

$\dot{y} = -10t + C$ $\quad \dot{x} = C$

At $t=0$ $80\sin\alpha = C$ $\quad$ At $t=0$ $80\cos\alpha = C$

$\therefore \dot{y} = -10t + 80\sin\alpha$ $\quad \dot{x} = 80\cos\alpha$

$y = -5t^2 + 80t\sin\alpha + C$ $\quad x = 80t\cos\alpha + C$

At $t=0$, $y=0$: $0 = C$ $\quad$ At $t=0$, $x=0$: $C=0$ $\quad x = 80t\cos\alpha$

$\therefore y = -5t^2 + 80t\sin\alpha$ $\quad x = 80t\cos\alpha$

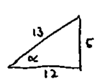
$s(t) = 80t\cos\alpha\,\underset{\sim}{i} + (80t\sin\alpha - 5t^2)\underset{\sim}{j}$

ii) $s_2(t)$ for particle 2

$s_2(t) = Vt\cos\beta\,\underset{\sim}{i} + (Vt\sin\beta - 5t^2)\underset{\sim}{j}$

Particles collide when $s_1(3)$ and $s_2(1)$

$\therefore V\cos\beta = 80\times 3\cos\alpha$ and $V\sin\beta - 5 = 80\times 3\sin\alpha - 45$



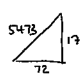
$V\cos\beta = 240\left(\frac{12}{13}\right)$ $\quad V\sin\beta = 240\times\frac{5}{13} - 40$

$= \frac{2880}{13}$ $\quad = \frac{680}{13}$

$\therefore \tan\beta = \frac{680}{2880}$

$= \frac{17}{72}$

iii) From i) $V\cos\beta = \frac{2880}{13}$



$V\left(\frac{72}{\sqrt{5473}}\right) = \frac{2880}{13}$

$V = \frac{2880}{13} \times \frac{\sqrt{5473}}{72}$

$= \frac{40}{13}\sqrt{5473}$

≈ 227.63 m/s (2dp)